Kiril Datchev
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## Homework 5

Due February 15th on paper at the beginning of class. Justify your answers. Please let me know if you have a question or find a mistake. There are some hints on the second page.

1. Borthwick Exercises 5.3, 5.4.
2. (a) Let $\Omega \subset \mathbb{R}^n$ be bounded, and let $0 < \lambda_1 \le \lambda_2 \le \cdots$ be the Dirichlet eigenvalues of $\Omega$, with corresponding eigenfunctions $\varphi_1, \varphi_2, \dots$. Given a constant $\gamma > 0$, consider the damped wave equation

$$u_{tt} + \gamma u_t - \Delta u = 0.$$

Find

$$\beta = \min\{\operatorname{Im}\omega\colon \text{ for some } k,\ u(t,x) = \varphi_k(x)e^{i\omega t} \text{ solves the damped wave equation.}\}$$

In words, $\beta$ is the smallest value $\operatorname{Im}\omega$ can take for damped waves.[1] The quantity $\beta$ is called the *decay rate* of the damped waves.

(b) Let $\mathcal{E}(t) = \mathcal{E}_K(t) + \mathcal{E}_P(t)$, where $\mathcal{E}_K(t) = \frac{1}{2}\int_\Omega |u_t|^2 dx$ and $\mathcal{E}_P(t) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$. Find a positive constant $C$ such that $\mathcal{E}'(t) = -C\gamma\mathcal{E}_K(t)$, and conclude that $\mathcal{E}(t)$ is nonincreasing.

(c) Let $n = 1$ and let $\Omega$ be an interval of lenght $L$. Find the choice of $\gamma = \gamma_c$ (in terms of $L$) that makes $\beta$ the largest it can be. (If $\gamma < \gamma_c$, then the system is *underdamped*, and more damping yields faster decay, but if $\gamma > \gamma_c$, then it is *overdamped*, and more damping yields slower decay; cf. the damped harmonic oscillator from ODEs.)

[1] At this point we are only examining damped waves of product form, but we will see that it works for more general damped waves.

*Hints:*

5.4. The values of $\omega$ are obtained by solving a quadratic equation, and as usual you get different cases depending on the sign of the discriminant.

2a. The value of $\beta$ should be in terms of $\gamma$ and $\lambda_1$, again with different cases depending on the sign of a discriminant.